Excerpted from R. A. Barnett, M. R. Ziegler, and K. E. Byleen, *Finite Mathematics for Business, Economics, Life Sciences, and Social Sciences*, 11th ed. (Upper Saddle River, NJ: Pearson / Prentice Hall, 2008), 417–418.

PROCEDURE **2 × 2 Matrix Games and Linear Programming: Geometric Approach**

Given the nonstrictly determined matrix game

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to find $P^* = [p_1 \quad p_2]$, $Q^* = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, and v, proceed as follows:

Step 1. If M is not a positive matrix (one with all entries positive), convert it into a positive matrix M_1 by adding a suitable positive constant k to each element. Let M_1, the new positive matrix, be represented as follows:

$$M_1 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad \begin{matrix} e = a + k & f = b + k \\ g = c + k & h = d + k \end{matrix}$$

This new matrix game, M_1, has the same optimal strategies P^* and Q^* as M. However, if v_1 is the value of the game M_1, then

corrected

$$v = v_1 - k$$

is the value of the original game M.

Step 2. Set up the two corresponding linear programming problems:

(A) Minimize $y = x_1 + x_2$
subject to $ex_1 + gx_2 \geq 1$
$fx_1 + hx_2 \geq 1$
$x_1, x_2 \geq 0$

(B) Maximize $y = z_1 + z_2$
subject to $ez_1 + fz_2 \leq 1$
$gz_1 + hz_2 \leq 1$
$z_1, z_2 \geq 0$

Step 3. Solve each linear programming problem geometrically.

Step 4. Use the solutions in step 3 to find the value v_1 for game M_1 and the optimal strategies and value v for the original game M:

$$v_1 = \frac{1}{y} = \frac{1}{x_1 + x_2} \quad \text{or} \quad v_1 = \frac{1}{y} = \frac{1}{z_1 + z_2}$$

$$P^* = [p_1 \quad p_2] = [v_1 x_1 \quad v_1 x_2] \qquad Q^* = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} v_1 z_1 \\ v_1 z_2 \end{bmatrix}$$

$$v = v_1 - k$$

Step 5. A further check of the solution is provided by showing that

$$P^* M Q^* = v \qquad \text{See Theorem 3, Section 10-2.}$$

INSIGHT

In step 2 in the box, note that part (B) is the dual of part (A), and consequently, both must have the same optimal value (Theorem 1, Section 6-3). In the next section, using the simplex method and properties of the dual, we will see that solving part (B) will automatically produce the solution for part (A). In this section we restrict our attention to the geometric approach; hence, we must solve each part as a separate problem.

EXAMPLE 1 **Solving 2 × 2 Matrix Games Using Geometric Methods** Solve the following matrix game using geometric methods to solve the corresponding linear programming problems (see Section 5-3):

$$M = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$$

SOLUTION **Step 1.** Convert M into a positive matrix (one with all entries positive) by adding 4 to each payoff. We denote the modified matrix by M_1:

$$M_1 = \begin{bmatrix} 2 & 8 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad k = 4$$

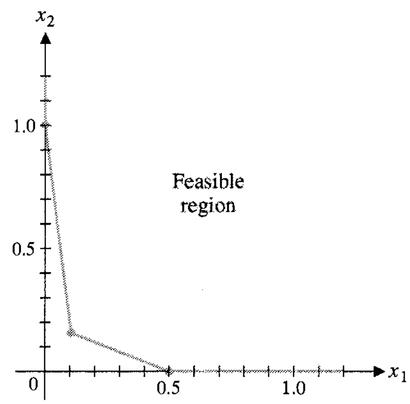
corrected

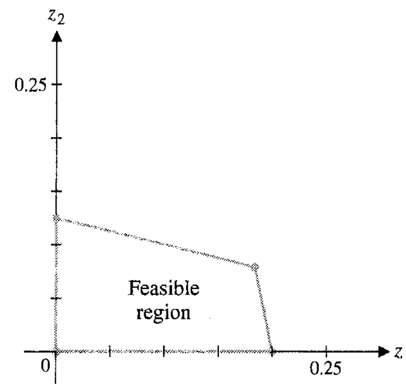
Step 2. Set up the two corresponding linear programming problems:

(A) Minimize $y = x_1 + x_2$
subject to
$$2x_1 + 5x_2 \ge 1$$
$$8x_1 + x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

(B) Maximize $y = z_1 + z_2$
subject to
$$2z_1 + 8z_2 \le 1$$
$$5z_1 + z_2 \le 1$$
$$z_1, z_2 \ge 0$$

Step 3. Solve each linear programming problem geometrically:





Theorems 1 and 2 in Section 5-3 imply that each problem has a solution that must occur at a corner point.

(A)

Corner Points	**Minimize** $y = x_1 + x_2$
$(0, 1)$	1
$(\frac{2}{19}, \frac{3}{19})$	$\frac{5}{19}$
$(\frac{1}{2}, 0)$	$\frac{1}{2}$

Min y occurs at

$x_1 = \frac{2}{19}$ and $x_2 = \frac{3}{19}$

(B)

Corner Points	**Maximize** $y = z_1 + z_2$
$(0, 0)$	0
$(0, \frac{1}{8})$	$\frac{1}{8}$
$(\frac{7}{38}, \frac{3}{38})$	$\frac{5}{19}$
$(\frac{1}{5}, 0)$	$\frac{1}{5}$

Max y occurs at

$z_1 = \frac{7}{38}$ and $z_2 = \frac{3}{38}$

Step 4. Use the solutions in step 3 to find the value v_1 for the game M_1 and the optimal strategies and value v for the original game M:

(A) $$v_1 = \frac{1}{x_1 + x_2} = \frac{1}{\frac{2}{19} + \frac{3}{19}} = \frac{19}{5}$$

$$p_1 = v_1 x_1 = \tfrac{19}{5} \cdot \tfrac{2}{19} = \tfrac{2}{5}$$
$$p_2 = v_1 x_2 = \tfrac{19}{5} \cdot \tfrac{3}{19} = \tfrac{3}{5}$$

(B) $$v_1 = \frac{1}{z_1 + z_2} = \frac{1}{\frac{7}{38} + \frac{3}{38}} = \frac{19}{5}$$

$$q_1 = v_1 z_1 = \tfrac{19}{5} \cdot \tfrac{7}{38} = \tfrac{7}{10}$$
$$q_2 = v_1 z_2 = \tfrac{19}{5} \cdot \tfrac{3}{38} = \tfrac{3}{10}$$

Note: v_1 found in part (A) should always be the same as v_1 found in part (B). Optimal strategies are the same for both games M and M_1. Thus,

$$P^* = [p_1 \quad p_2] = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix} \qquad Q^* = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{10} \\ \frac{3}{10} \end{bmatrix}$$

and the value of the original game is

$$v = v_1 - k = \tfrac{19}{5} - 4 = -\tfrac{1}{5}$$

Step 5. A further check of the solution is provided by showing that

$$P^*MQ^* = v \qquad \text{See Theorem 3, Section 10-2.}$$

This check is left to the reader. ■

MATCHED PROBLEM 1 Solve the following matrix game using geometric linear programming methods:

$$M = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$